

# HEIGHT ESTIMATES FOR DOMINANT ENDOMORPHISMS ON PROJECTIVE VARIETIES

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ABSTRACT. If  $\phi$  is a polarizable endomorphism on a projective variety, then the Weil height function gives a relation between the height of a point and the height of its image under  $\phi$ . In this paper, we generalize this result to arbitrary dominant endomorphisms. We define height expansion and contraction coefficients for dominant morphisms, compare these to Silverman's height expansion coefficient in [16], and provide several examples of dynamical systems on projective varieties.

## 1. INTRODUCTION

A dynamical system consists of a set  $S$  and a map  $\phi : S \rightarrow S$  maps  $S$  to  $S$  itself. Thus, more structure  $S$  has, more dynamical information we gain. When  $S$  is a projective variety, it has the Weil height functions so that arithmetic dynamics gains lots of information from them. Moreover, if we have a special kind of morphism, then we have pleasant result: we say that  $\phi$  is *polarizable* if there is an ample divisor  $D \in \text{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R}$  such that  $\phi^*D$  is linearly equivalent to  $q \cdot D$  where  $q$  is a positive real number. If  $\phi$  is a polarizable defined over a number field  $K$ , then it satisfies the Northcott's property: we say  $\phi$  *satisfies the Northcott's property* if the following equality holds for some Weil height function  $h_D$  corresponding an ample divisor  $D$ :

$$h_D(\phi(P)) = q \cdot h_D(P) + O(1) \quad \text{for all } P \in S(\overline{K}).$$

If  $\phi$  is not polarizable, then it does not satisfy the Northcott's property. For example, an automorphism of infinite order on  $K3$  surface is not polarizable so that we can't expect the above height inequality. However, we can still expect to find the relation between the height values of points  $P$  and  $\phi(P)$ . We say that  $\phi$  *satisfies the weak Northcott's property* if there are a Weil height  $h_D$  corresponding an ample divisor and two constants  $C_1, C_2$  such that

$$C_1 \cdot h_D(\phi(P)) - O(1) \leq h_D(P) \leq C_2 \cdot h_D(\phi(P)) + O(1).$$

The main purpose of this paper is that every 'dominant' endomorphism satisfies the weak Northcott's property. In section 2, Every dominant endomorphisms generates a map on the ample cone. This fact allows us to find the constants for the weak Northcott's property. (Well-definedness is guaranteed by Lemma 3.1.)

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*Date:* October 28, 2010.

*1991 Mathematics Subject Classification.* Primary: 37P30 Secondary: 11G50, 32H50, 37P05.

*Key words and phrases.* height, dominant morphism, preperiodic points, Northcott's property.

**Definition 1.1.** Let  $W$  be a projective variety and let  $\phi : W \rightarrow W$  be a dominant morphism. We define the height expansion coefficient of  $\phi$  for  $D$

$$\mu_1(\phi, D) := \sup\{\alpha \in \mathbb{R} \mid \phi^*D - \alpha D \text{ is ample}\}$$

and the height contraction coefficient of  $\phi$  for  $D$

$$\mu_2(\phi, D) := \inf\{\alpha \in \mathbb{R} \mid \alpha D - \phi^*D \text{ is ample}\}.$$

**Theorem 1.2.** Let  $W$  be a projective variety, let  $\phi : W \rightarrow W$  be a dominant endomorphism defined over a number field  $K$ , let  $D$  be an ample divisor on  $W$  and  $\mu_1 = \mu_1(\phi, D), \mu_2 = \mu_2(\phi, D)$  be the height expansion and contraction coefficients of  $\phi$  for  $D$ . Then, for any  $\epsilon > 0$ , there are constants  $C_1, C_2$  satisfying

$$\frac{1}{\mu_1 - \epsilon} h_D(\phi(P)) + C_1 \geq h_D(P) \geq \frac{1}{\mu_2 + \epsilon} h_D(P) - C_2$$

for all  $P \in W(\overline{K})$ .

Interestingly, we have Silverman's height expansion coefficient defined on [16]: a dominant endomorphism is clearly an example of a equidimensional dominant rational map. In section 3, we will show that they are the Silverman's height expansion coefficient is the same with  $\mu_1$ ;

**Proposition 1.3.** Let  $\phi : W \rightarrow W$  be a dominant morphism defined over a number field, let  $D$  be an ample divisor and let  $\mu_1(\phi, D)$  be the height expansion coefficient of  $\phi$ . Then,

$$\mu_1(\phi, D) = \liminf_{h_D(P) \rightarrow \infty} \frac{h_D(\phi(P))}{h_D(P)}.$$

From now on, we will let  $W$  be a projective variety, let  $\phi : W \rightarrow W$  be a dominant endomorphism on  $W$  defined over a number field  $K$  and let  $D$  be an ample divisor on  $W$  unless state otherwise.  
*Acknowledgements.* The author would like to thank Joseph H. Silverman and Dan Abramovich for their helpful advice and comments.

## 2. DOMINANT ENDOMORPHISM AND PULL-BACKS OF AMPLE DIVISORS

To satisfy the weak Northcott's property,  $\phi$  should be at least quasi-finite: suppose not. Then we have a point  $P$  whose inverse image is a subvariety  $Y$ . Thus,  $h_D(P)$  is constant while  $h_D(Q)$  goes to infinity on  $Y$ . Usually, a dominant morphism need not be quasi-finite. However, for endomorphism on a projective variety, 'quasi-finiteness' condition is equivalent to 'dominance' condition.

**Definition 2.1.** Let  $\psi : W \rightarrow V$  be a rational map. We say that  $\psi$  is dominant if  $\overline{\psi(W)} = V$ .

**Proposition 2.2.** Let  $\phi : W \rightarrow W$  be an endomorphism. Then The followings are equivalent;

- (1)  $\phi$  is dominant.
- (2)  $\phi$  is quasi-finite.
- (3)  $\phi$  is finite.

*Proof.* (1)  $\Rightarrow$  (3) Since  $W$  is a projective variety,  $W$  is compact and hence  $\phi$  is surjective. Then, [11, §4] says that surjective holomorphic endomorphism on a projective variety is finite.

(3)  $\Rightarrow$  (1) It is a property of finite morphism; if  $\phi$  is not dominant, then  $\phi$  is not quasi-finite and hence not finite.

(2)  $\Leftrightarrow$  (3) [3, §8.11.1] says that  $\phi$  is finite if  $\phi$  is proper, locally of finite presentation and quasi-finite. Since  $W$  is a projective variety,  $\phi$  is automatically projective and hence proper and locally of finite presentation. Therefore, if  $\phi$  is quasi-finite, then  $\phi$  is finite.  $\square$

Let  $\phi$  be defined over a number field. To study the Weil height function value of the image of some morphism  $h_D(\phi(P))$ , it is essential to observe  $\phi^*D$  because of the functorial property of the Weil height machine:

$$h_D(\phi(P)) = h_{\phi^*D}(P) + O(1).$$

If  $\phi : W \rightarrow W$  is a polarizable, then, by definition, there is an ample divisor  $E$  such that  $q \cdot D \sim \phi^*E$ , which implies that  $\phi^*E$  is ample. It is also true for general dominant endomorphism because  $\phi$  is quasi-finite.

**Proposition 2.3.** *Let  $\phi : W \rightarrow W$  be a morphism. Then, the followings are equivalent;*

- (1)  $\phi$  is dominant.
- (2)  $\phi^*E$  is ample for some ample divisor  $E$ .
- (3)  $\phi^*E$  is ample for all ample divisors  $E$ .

*Proof.* (1)  $\Rightarrow$  (3) Suppose that  $\phi^*E$  is not ample for an ample divisor  $E$ . Then, By Kleiman's criterion, there is a pseudo-effective 1-cycle  $C$  (limit of effective cycle) such that  $C \cdot \phi^*E \leq 0$ . More precisely, since  $E$  is ample and hence numerically effective,  $\phi^*E$  is also numerically effective and hence  $C \cdot \phi^*E = 0$ . Because of projection formula for intersection, we have

$$\phi_*C \cdot E = C \cdot \phi^*E = 0.$$

If  $\phi_*C$  is pseudo-effective 1-cycle, then  $\phi_*C \cdot E > 0$  because of Kleiman's Criterion again. It is contradiction so that  $\phi(C)$  should be a zero-cycle and hence numerically equivalent to finite sum of points. However,  $\phi$  is dominant and hence is quasi-finite. Therefore, the preimage of a finite set of points is a finite set of points again, so we again have a contradiction.

(3)  $\Rightarrow$  (2) It is trivial.

(2)  $\Rightarrow$  (3) Let  $D$  be an ample divisor on  $W$  such that  $\phi^*D$  is ample. Suppose that there is an ample divisor  $E$  such that  $\phi^*E$  is not ample. Then, by Nakai-Moishezon Criterion, there is an integral subvariety  $Y \subset W$  of dimension  $r$  such that

$$(\phi^*E)^r \cdot Y \leq 0.$$

Then, by the projection formula for intersection, we have

$$E^r \cdot \phi_*Y = \phi^*(E^r) \cdot Y = (\phi^*E)^r \cdot Y \leq 0.$$

If  $\phi_*Y$  is a subvariety of dimension  $r$ , then it is contradiction because  $E$  is an ample divisor. So,  $\phi_*Y$  should be of dimension  $r' < r$ . However, since  $\phi^*D$  is ample,

$$0 < (\phi^*D)^r \cdot Y = D^r \cdot \phi_*Y = 0$$

and hence it is also contradiction. Therefore,  $\phi^*E$  is also ample.

(3)  $\Rightarrow$  (1) If  $\phi$  is not dominant, then  $\dim \phi(W) < \dim W$ . So, there is a subvariety  $V \subset W$  such that  $\phi(V) = Q \in W$ . Therefore, for an ample divisor  $E$ ,

$$h_{\phi^*E}(P) = h_E(\phi(P)) + O(1) = h_E(Q) + O(1) \quad \text{for all } P \in V.$$

Thus, the height corresponding  $\phi^*E$  is bounded on a variety  $V$  and hence  $\phi^*E$  is not ample.  $\square$

### 3. THE HEIGHT EXPANSION AND CONTRACTION CONSTANT

In this section, we will define the height expansion and contraction coefficients and will build the height inequality. It starts from a basic property of ample divisors:

**Lemma 3.1.** *Let  $W$  be a projective variety and  $D_1, D_2$  be ample divisors on  $W$ . Then, there is a positive constant  $\alpha$  such that  $\alpha D_1 - D_2$  is ample again.*

*Proof.* [14, Theorem A.3.2.3] or [4].  $\square$

The Lemma 3.1 guarantees the well-definedness of Definition 1.1 since  $\{\alpha \in \mathbb{R} \mid \phi^*D - \alpha D \text{ is ample}\}$  is not an empty set. Once well defined, the height expansion and contraction coefficients will provide the weak Northcott's property;

**Theorem 1.2.** *Let  $\phi : W \rightarrow W$  be a dominant endomorphism defined over a number field  $K$ , let  $D$  be an ample divisor on  $W$  and  $\mu_1 = \mu_1(\phi, D), \mu_2 = \mu_2(\phi, D)$  be the height expansion and contraction coefficients of  $\phi$  for  $D$ . Then, for any  $\epsilon > 0$ , there are constants  $C_1, C_2$  satisfying*

$$\frac{1}{\mu_1 - \epsilon} h_D(\phi(P)) + C_1 \geq h_D(P) \geq \frac{1}{\mu_2 + \epsilon} h_D(P) - C_2$$

for all  $P \in W(\overline{K})$ .

*Proof.* Then, for any  $\epsilon > 0$ , both  $E_1 = \phi^*D - (\mu_1 - \epsilon)D$  and  $E_2 = (\mu_2 + \epsilon)D - \phi^*D$  are ample. Thus,  $h_{E_1}$  and  $h_{E_2}$  are bounded below. Therefore,

$$h_D(\phi(P)) - (\mu_1 - \epsilon)h_D(P) = h_{E_1}(P) + O(1) > O(1)$$

and

$$(\mu_2 + \epsilon)h_D(P) - h_D(\phi(P)) = h_{E_2}(P) + O(1) > O(1).$$

Finally,

$$\frac{1}{\mu_1 - \epsilon} h_D(\phi(P)) + C_1 \geq h_D(P) \geq \frac{1}{\mu_2 + \epsilon} h_D(P) - C_2.$$

$\square$

**Remark 3.2.** We may expect the following inequality:

$$\frac{1}{\mu_1}h_D(\phi(P)) + C_1 \geq h_D(P) \geq \frac{1}{\mu_2}h_D(P) - C_2.$$

Unfortunately, it may not be true because  $\phi^*D - \mu_1D$  and  $\mu_2\phi^*D - D$  are just numerically effective divisors so that the Weil heights corresponding to those divisors may not be bounded below on entire  $W$ .

For example, Let  $W$  be an elliptic curve and let  $\phi = [N]$ . Choose a point  $P$  and let  $Q = [N](P)$ . Then, the divisor  $q(P) - (Q)$  is ample if and only if  $q > 1$  and hence  $\mu_1([N], (P)) = 1$ . However,  $\hat{h}_Q(R) - \hat{h}_P(R) = \hat{h}_{Q-P}(R) = 2\langle Q - P, R \rangle$  may go to  $-\infty$ .

**Example 3.3.** Suppose that  $\phi$  is a polarizable morphism with respect to an ample divisor  $D$ :

$$\phi^*D \sim q \cdot D.$$

Then,  $\mu_1(\phi, D) = \mu_2(\phi, D) = q$  and hence it satisfies the Northcott's property.

**Example 3.4.** Let  $V \subset \mathbb{P}^2 \times \mathbb{P}^2$  be a K3-surface and let  $\iota_1, \iota_2$  be involutions on  $V$ . Let  $D_1, D_2$  be pullbacks of  $H \times \mathbb{P}^2$  and  $\mathbb{P}^2 \times H$  and  $E_+ = -D_1 + \beta D_2$ ,  $E_- = D_2 + \beta^{-1}D_1$  where  $\beta = 2 + \sqrt{3}$ . Then, divisor  $D = aE_+ + bE_-$  is ample if and only if  $a, b > 0$ .

Then,  $\iota_1^*(aE_+ + bE_-) = \beta(aE_-) + \beta^{-1}(bE_+)$ . Thus,

$$\begin{aligned} \mu_1(\iota_1, E_+ + E_-) &= \sup\{\alpha \mid \beta^{-1} - \alpha > 0, \beta - \alpha > 0\} \\ &= \min(\beta^{-1}, \beta) \\ &= \beta^{-1}. \end{aligned}$$

Let  $\phi = \iota_2 \circ \iota_1$ . Then, it is dominant because

$$\phi^*(aE_+ + bE_-) = \iota_1^*(\iota_2^*(aE_+ + bE_-)) = \iota_1^*(\beta aE_- + \beta^{-1}bE_+) = \beta^{-2}aE_+ + \beta^2bE_-.$$

Thus,

$$\phi^*(aE_+ + bE_-) - \alpha(aE_+ + bE_-) = a(\beta^{-2} - \alpha)E_+ + b(\beta^2 - \alpha)E_-.$$

Therefore,  $\mu_1(\phi, aE_+ + bE_-) = \beta^{-2}$  and hence  $\mu(\phi) = \beta^{-2}$ . Similarly,  $\mu_2(\phi, aE_+ + bE_-) = \beta^2$ .

**Example 3.5.** Let  $V \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  be a generic hypersurface of tridegrees  $(2, 2, 2)$ . Let  $\iota_1, \iota_2$  and  $\iota_3$  be involutions on  $V$ . Then, the ample cone is the light cone

$$\mathcal{L}^+ = \{E \in \text{Pic}(V) \mid E^2 > 0, E \cdot D_0 > 0\}$$

where  $D_0$  is arbitrary ample divisor. Let  $E_i$  be pullbacks of hyperplane  $H_i$  of  $i$ -th component. Since the Picard number of  $V$  is three,  $\{E_1, E_2, E_3\}$  is a generator of  $\text{Pic}(V)$ . Moreover,  $E_a = E_1 + E_2 + E_3$  is very ample divisor corresponding Segre embedding and the intersection number of  $\{E_1, E_2, E_3\}$  is

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

Therefore, the ample cone is described with the coefficient:

$$\left\{ \sum a_i E_i \mid \sum_{i \neq j} a_i a_j > 0, \sum a_i > 0 \right\}.$$

Then,  $\iota_1^* D = -a_1 E_1 + (2a_1 + a_2) E_2 + (2a_1 + a_3) E_3$ . Thus,

$$\begin{aligned} \mu_1(\iota_1, E_a) &= \sup\{\alpha \mid (-1 - \alpha) E_1 + (3 - \alpha) E_2 + (3 - \alpha) E_3 : \text{ample}\} \\ &= \sup\{\alpha \mid (5 - 3\alpha) > 0, (\alpha - 3)(3\alpha - 1) > 0\} \\ &= \frac{1}{3}. \end{aligned}$$

and

$$\begin{aligned} \mu_2(\iota_1, E_a) &= \inf\{\alpha \mid (\alpha + 1) E_1 + (\alpha - 3) E_2 + (\alpha - 3) E_3 : \text{ample}\} \\ &= \inf\{\alpha \mid (3\alpha - 5) > 0, (\alpha - 3)(3\alpha - 1) > 0\} \\ &= 3. \end{aligned}$$

Let  $\phi_{1,2} = \iota_2 \circ \iota_1$ . Then, it is dominant because

$$\phi_{1,2}^* E_a = \iota_1^*(\iota_2^* E_a) = \iota_1^*(3E_1 - E_2 + 3E_3) = -3E_1 + 5E_2 + 9E_3.$$

Thus,

$$\begin{aligned} \mu_1(\phi_{1,2}, E_a) &= \sup\{\alpha \mid (-3 - \alpha) E_1 + (5 - \alpha) E_2 + (9 - \alpha) E_3 : \text{ample}\} \\ &= \sup\{\alpha \mid (11 - 3\alpha) > 0, 3\alpha^2 - 22\alpha + 3 > 0\} \\ &= \frac{11 - \sqrt{112}}{3}. \end{aligned}$$

**Example 3.6.** Let  $\mathbb{X} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_k}$  where  $n_i < n_{i+1}$  and let  $\phi$  be a dominant endomorphism of  $\mathbb{X}$  defined over a number field. Then, by Appendix A,  $\phi = (\phi_1, \dots, \phi_k)$  where  $\phi_i : \mathbb{P}^{n_i} \rightarrow \mathbb{P}^{n_i}$  is a morphism on projective space. Let  $\pi_i : \mathbb{X} \rightarrow \mathbb{P}^{n_i}$  be a projection map, let  $\iota_i : \mathbb{P}^{n_i} \rightarrow \mathbb{X}$  be a closed embedding map and let  $E_i = \pi_i^* H_i$  where  $H_i$  is a hyperplane of  $\mathbb{P}^{n_i}$ . Then, a divisor  $D = \sum_i 1^k a_i E_i$  is ample if and only if  $a_i > 0$  for all  $i$ . Furthermore,  $\phi^* E_i = \deg \phi_i \cdot E_i$  and hence

$$\mu_1(\phi, D) = \min \deg \phi_i \quad \mu_2(\phi, D) = \max \deg \phi_i.$$

#### 4. SILVERMAN'S HEIGHT EXPANSION COEFFICIENT

Silverman [16] introduced the height expansion coefficient for equidimensional dominant rational maps;

**Definition 4.1.** Let  $\psi : W \dashrightarrow V$  be a dominant rational map between quasiprojective varieties with the same dimension, all defined over  $\overline{\mathbb{Q}}$ . Fix height functions  $h_{D_V}$  and  $h_{D_W}$  on  $V$  and  $W$

respectively, corresponding to ample divisors  $D_V$  and  $D_W$ . The height expansion coefficient of  $\psi$  (relative to chosen ample divisors  $D_V$  and  $D_W$ ) is the quantity

$$\mu'(\psi, D_W, D_V) = \sup_{\emptyset \neq U \subset W} \liminf_{P \in U(\overline{\mathbb{Q}})} \frac{h_{D_V}(\psi(P))}{h_{D_W}(P)},$$

where the sup is over all nonempty Zariski dense open subsets of  $W$ .

Then, the following theorem shows the relation between Definition 1.1 and Definition 4.1

*Proof of Proposition 1.3.* For dominant endomorphism  $\phi : W \rightarrow W$ ,  $\phi$  is defined on entire  $W$ . Thus, the supremum comes from the biggest open set of  $W$ , which is  $W$  itself:

$$\mu'(\phi, D, D) = \sup_{\emptyset \neq U \subset W} \liminf_{h_D(P) \rightarrow \infty} \frac{h_D(\phi(P))}{h_D(P)} = \liminf_{h_D(P) \rightarrow \infty} \frac{h_D(\phi(P))}{h_D(P)}.$$

Let  $\mu_1 = \mu_1(\phi, D)$  and  $\epsilon > 0$  be any positive number. Then, there is a  $\delta \in [0, \epsilon]$  such that  $\phi^*D - (\mu_1 - \delta)D$  is ample. Thus,

$$h_{\phi^*D}(P) - (\mu_1 - \delta)h_D(P) \geq O(1).$$

Therefore,

$$(1) \quad \frac{h_{\phi^*D}(P) - O(1)}{h_D(P)} \geq \mu_1 - \delta \quad \text{and} \quad \liminf_{h_D(P) \rightarrow \infty} \frac{h_{\phi^*D}(P)}{h_D(P)} \geq \mu_1 - \delta \geq \mu_1 - \epsilon.$$

On the other hand, let  $E = \phi^*D - (\mu_1 + \epsilon)D$ . Then, there is an irreducible curve  $C$  such that  $E \cdot C < 0$ ; otherwise, then  $E$  is a numerically effective divisor so that  $E + \frac{\epsilon}{2}D$  is ample. But, it contradicts to the definition of  $\mu_1$ .

Then, we have

$$\lim_{\substack{h_D(P) \rightarrow \infty \\ P \in C}} \frac{h_E(P)}{h_D(P)} = \frac{E \cdot C}{D \cdot C} < 0$$

and hence

$$\liminf_{h_D(P) \rightarrow \infty} \frac{h_E(P)}{h_D(P)} \leq \lim_{\substack{h_D(P) \rightarrow \infty \\ P \in C}} \frac{h_E(P)}{h_D(P)} < 0.$$

So,

$$(2) \quad \liminf_{h_D(P) \rightarrow \infty} \frac{h_{\phi^*D}(P)}{h_D(P)} < \liminf_{h_D(P) \rightarrow \infty} \frac{h_{(\mu_1 + \epsilon)D}(P)}{h_D(P)} = \mu_1 + \epsilon.$$

Combine (1) and (2) and get

$$\mu_1 - \epsilon \leq \liminf_{h_D(P) \rightarrow \infty} \frac{h_D(\phi(P))}{h_D(P)} \leq \mu_1 + \epsilon$$

for any  $\epsilon > 0$ . Therefore, we get the desired result.  $\square$

## 5. APPLICATIONS

**5.1. arithmetic dynamics.** The height expansion coefficient has an application in arithmetic dynamics. We know that  $\text{Preper}(\phi)$  is of bounded height when  $\phi$  is polarizable with  $q > 1$ . Recall that  $q = \mu_1(\phi, D)$ . Thus, it is not weird to expect the similar result for dominant endomorphism with the height expansion coefficient.

**Definition 5.1.** Let  $\phi : W(K) \rightarrow W(K)$  be a dominant morphism defined over a number field  $K$ . We define the global height expansion coefficient of  $\phi$ :

$$\mu(\phi) = \sup_{D: \text{ ample}} \mu_1(\phi, D).$$

**Theorem 5.2.** Let  $\phi : W \rightarrow W$  be a dominant endomorphism and  $E$  be an ample divisor. Suppose that the global height expansion coefficient  $\mu(\phi) > 1$ . Then, the set of preperiodic points is of bounded height by  $h_E$ .

*Proof.* Let  $\mu(\phi) > 1$ . Then, there is an ample divisor  $D$  such that  $\mu_1(\phi, D) > 1$ . Suppose that  $\epsilon = \frac{\mu_1(\phi, D) - 1}{2}$ . Then,

$$\frac{1}{\mu_1(\phi, D) - \epsilon} h_D(\phi(P)) = \frac{1}{1 + \epsilon} h_D(\phi(P)) \geq h_D(P) - C.$$

By telescoping sum, we have

$$\lim_{n \rightarrow \infty} \left( \frac{1}{1 + \epsilon} \right)^n h_D(\phi^n(P)) \geq h_D(P) - \frac{1}{1 - \frac{1}{1 + \epsilon}} C.$$

Therefore, if  $P \in \text{Preper}(\phi)$ , then the left hand side goes to zero so that  $h_D(P)$  is bounded.

Moreover, if  $E$  is another ample divisor then Lemma 3.1 says that  $\alpha \cdot D - E$  is ample for sufficiently large  $\alpha > 0$ . Since the Weil height corresponding the ample divisor is bounded below and hence

$$\alpha \cdot h_D(P) + O(1) > h_E(P)$$

for all  $P \in W$ . Therefore,  $h_E(\text{Pre}(\phi))$  is also bounded.  $\square$

**Example 5.3.** Consider the very first example; let  $f_i : \mathbb{P}^n \rightarrow \mathbb{P}^n$  be a morphism of degree  $d_i > 1$ . Then, a morphism

$$\phi = \prod f_i : (\mathbb{P}^n)^m \rightarrow (\mathbb{P}^n)^m$$

is a dominant morphism of  $\mu(\phi) = \min d_i > 1$ . Thus,  $\text{Preper}(\phi)$  is a set of bounded height. Precise calculation appears on Appendix A.

**5.2. Seshadri Constant.** The height expansion coefficient has a relation with the Seshadri constant. Demailly [2] defined the Seshadri constant.

**Definition 5.4.** Let  $Y$  be a closed subscheme of  $X$  whose underlying subvariety is of codimension  $r > 1$ , let  $\tilde{X}$  be a blowup of  $X$  along  $Y$  and let  $L$  be a numerically effective divisor of  $X$ . Then, we define the generalized Seshadri constant

$$\epsilon(L, Y) = \sup\{\alpha \mid \pi^* L - \alpha E : \text{ numerically effective}\}.$$



Similarly, we define the  $s$ -invariant

$$s_L(Y) = \min\{s \mid s \cdot \pi^*L - E : \text{numerically effective}\}.$$

**Theorem 5.5.** *Let  $\phi : W \rightarrow W$  be a dominant morphism and let  $D$  be a ample divisor. Then,*

$$\epsilon(\phi^*D, D) \geq \mu_1(\phi, D).$$

*Proof.* The ample cone of  $V$  is a subcone of the nef cone and hence

$$\begin{aligned} \mu_1(\phi, D) &= \sup\{\alpha \mid \tilde{\phi}^*D - \alpha D \text{ is ample.}\} \\ &= \sup\{\alpha \mid \tilde{\phi}^*D - \alpha D \text{ is numerically effective.}\} \\ &= \epsilon(\phi^*D, D). \end{aligned}$$

□

#### APPENDIX A. EXAMPLE: DOMINANT MORPHISMS ON $\mathbb{X} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$

In this section, we will show that dominant endomorphisms on  $\mathbb{X} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$  is a block diagonal one. So, we only have to treat Example 5.3 in the view of arithmetic dynamics because for any dominant endomorphism  $\phi$  on  $\mathbb{X}$ , there is a integer  $N$  such that  $\phi^N$  is a Cartesian product of endomorphisms  $\psi_i : \mathbb{P}^{n_i} \rightarrow \mathbb{P}^{n_i}$ .

**A.1. Basic notations for morphisms on  $\mathbb{X}$ .** Let  $H_i$  be a hyperplane of  $\mathbb{P}^{n_i}$  which generate  $\text{Pic}(\mathbb{P}^{n_i})$  and  $\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \rightarrow \mathbb{P}^{n_i}$  be a  $i$ -th projection map. Let  $E_i = \pi_i^*H_i$ . Then,  $\text{Pic}(\mathbb{X}) = \langle E_1, \dots, E_k \rangle$ . Let  $\mathfrak{X} = \text{Pic}(\mathbb{X}) \otimes \mathbb{R} = \sum \mathbb{R}E_i$  and consider  $\phi^*$  as matrix with basis  $\{E_1, \dots, E_k\}$ .

**Lemma A.1.** *Let  $D = \sum a_i E_i \in \text{Pic}(\mathbb{X})$ . Then  $D$  is ample if and only if  $a_i > 0$  for all  $i$ .*

*Proof.* Clearly  $D_0 = \sum_{i=1}^k E_i$  is ample; consider the Segre embedding

$$\tau : \mathbb{X} \rightarrow \mathbb{P}^N.$$

Then,  $D_0 = \tau^*H_{\mathbb{P}^N}$  where  $H_{\mathbb{P}^N}$  is a hyperplane on  $\mathbb{P}^N$ .

For general  $D$ , we can find  $\beta > 0$  such that

$$D = \beta \cdot D_0 + \sum \gamma_i E_i \quad \text{where } \gamma_i \geq 0.$$

Then,  $D_0$  is ample and  $\sum \gamma_i E_i$  is nef. Thus the sum of these two divisors is ample. □

**A.2. Morphisms  $f : \mathbb{X} \rightarrow \mathbb{P}^m$ .** In this subsection, we will study the morphism  $f : \mathbb{X} \rightarrow \mathbb{P}^m$  which will be a component of endomorphism  $\phi$  on  $\mathbb{X}$ . For the convenience, assume that it's sorted by dimension:  $n_i \leq n_{i+1}$ .

**Lemma A.2.** *Let  $f : \mathbb{X} \rightarrow \mathbb{P}^m$  be a morphism and*

$$f^* : \text{Pic}(\mathbb{P}^m) \otimes \mathbb{R} = \mathbb{R} \rightarrow \text{Pic}(\mathbb{X}) = \mathbb{R}^k, \quad 1 \mapsto (d_1, \dots, d_k)$$

*Suppose  $\sum_{i=1}^k (n_i + 1) > m + 1$ . Then, there is an index  $j$  such that  $d_j = 0$ .*

*Proof.* Let  $A^*X$  be a Chow ring of a projective variety  $X$ . Then, [5, Example 8.3.4] says that

$$A^*X \otimes A^*\mathbb{P}^l \simeq A^*(X \times \mathbb{P}^l).$$

Therefore,

$$A^r(\mathbb{X}) = \bigotimes_{\sum r_i=r} A^{r_i}(\mathbb{P}^{n_i}).$$

Let  $H$  be a hyperplane on  $\mathbb{P}^m$ . Because  $f^*$  is ring homomorphism on the Chow ring and  $f^*H \sim \sum_{i=1}^k d_i E_i$ ,  $(f^*H)^{m+1} = (\sum_{i=1}^k d_i E_i)^{m+1}$  where  $H^{m+1}$  is  $(m+1)$ -th self intersection of  $H$ . Furthermore,  $H^{m+1} = 0$  so that

$$0 = (\phi^*H)^{m+1} = \left( \sum_{i=1}^k d_i E_i \right)^{m+1} = \sum_I C_I E_1^{\alpha_{I1}} \cdots E_k^{\alpha_{Ik}}$$

where  $C_I = \binom{\alpha_1, \dots, \alpha_k}{m+1} d_1^{\alpha_1} \cdots d_k^{\alpha_k}$  are positive integers. However, the assumption  $\sum_{i=1}^k (n_i + 1) > m + 1$  does not allow middle parts to vanish;  $E_1^{\alpha_1} \cdots E_k^{\alpha_k} \neq 0$  if  $\alpha_i \leq n_i$  for all  $i = 1, \dots, k$ . Furthermore,  $E_1^{\alpha_1} \cdots E_k^{\alpha_k}$  are linearly independent. Therefore,  $d_j = 0$  for some  $j$ .  $\square$

**Corollary A.3.** *Let*

$$f : \mathbb{X} \rightarrow \mathbb{P}^m$$

*be a morphism. Suppose  $f^* = (d_1, \dots, d_k)$  ( $f^*H = \sum_{i=1}^k d_i E_i$ ). Then,  $d_i = 0$  for all  $i$  satisfying  $n_i > m$ .*

*Proof.* Let  $\iota_i : \mathbb{P}^{n_1} \rightarrow \mathbb{X}$  be a closed embedding and let  $H$  be a hyperplane on  $\mathbb{P}^m$ . Then,  $f \circ \iota_i : \mathbb{P}^{n_i} \rightarrow \mathbb{P}^m$  is a morphism such that  $(f \circ \iota_i)^*H = d_i H_{\mathbb{P}^{n_i}}$ . If  $n_i > m$ , then  $d_i = 0$  because of the Lemma A.2.  $\square$

**Theorem A.4.** *Let*

$$f : \mathbb{X} \rightarrow \mathbb{P}^m$$

*be a morphism. If  $\sum_{i=1}^k (n_i + 1) > m + 1$ , then we can forget a factor of  $\mathbb{X} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ ; there is a map*

$$g : \mathbb{X}' = \prod_{j \in J} \mathbb{P}^{n_j} \rightarrow \mathbb{P}^m$$

*where  $J \subsetneq \{1, \dots, k\}$  such that*

$$\begin{array}{ccc} \mathbb{X} & & \\ \pi \downarrow & \searrow f & \\ \mathbb{X}' & \xrightarrow{g} & \mathbb{P}^m \end{array}$$

*Moreover, we can claim  $\sum_{j \in J} (n_j + 1) \leq m + 1$ .*

*Proof.* Suppose the assumption is true. Then, by Lemma A.2,  $d_j = 0$  for some  $i$ . Thus, we may assume that  $\phi$  is a constant morphism in terms of variables  $X_j$  on  $\mathbb{P}^{n_j}$ . Therefore, we can consider

$$g : (\mathbb{P}^{n_1} \times \cdots \widehat{\mathbb{P}^{n_j}} \cdots \times \mathbb{P}^{n_k}) \rightarrow \mathbb{P}^m.$$

If  $\sum_{i \neq j} (n_i + 1) > m + 1$ , then we can apply Lemma A.2 again.  $\square$

**A.3. Endomorphisms on  $\mathbb{Y} = (\mathbb{P}^n)^l$ .** In the next subsection, we will have the dominant endomorphism on  $\mathbb{X}$  is a product of endomorphism on  $(\mathbb{P}^n)^l$ . Thus, we will check the endomorphism on  $(\mathbb{P}^n)^l$  to prepare the final result.

Let  $\mathbb{Y} = (\mathbb{P}^n)^l$  and  $\mathfrak{Y} = \text{Pic}(\mathbb{Y}) \otimes \mathbb{R} = \mathbb{R}E_1 \oplus \cdots \oplus \mathbb{R}E_l$ .

**Lemma A.5.** *Let*

$$\psi(P) =: \mathbb{Y} \longrightarrow \mathbb{Y}$$

*be a morphism. Then  $\psi$  only depends on one of  $\mathbb{P}^n$ .*

*Proof.* Let  $\psi^* = (d_1, \dots, d_l)$ . Then, since  $n + 1 < 2n + 2$ , Theorem A.4 tells that exactly one of  $d_i$  can be nonzero.  $\square$

**Corollary A.6.** *Let  $\psi : \mathbb{Y} \rightarrow \mathbb{Y}$  be a dominant endomorphism. Then*

$$\psi^* : \mathfrak{Y} \rightarrow \mathfrak{Y}$$

*is a  $l \times l$ -matrix such that there is only one nonzero element on each row;*

$$\psi^* = \begin{pmatrix} d_1 \mathbf{e}_{\sigma(1)} \\ \vdots \\ d_l \mathbf{e}_{\sigma(l)} \end{pmatrix}$$

*where  $\mathbf{e}_j$  is  $j$ -th elementary row vector and  $\sigma \in S_l$  is a permutation map defined by  $\psi^* E_i = d_i E_{\sigma(i)}$ .*

*Proof.* Let  $\psi = (\psi_1, \dots, \psi_l)$ . Lemma A.5 says that  $\psi_j^*$  is a row vector whose elements are zero except one. If  $\psi_u^*$  and  $\psi_v^*$  has nonzero element on the same column, then there is a zero column on  $\psi^*$  and hence  $\psi^* E$  is not ample for any ample divisor  $E$ . It contradicts to  $\psi$  is dominant.  $\square$

**Corollary A.7.** *Let  $\psi : \mathbb{Y} \rightarrow \mathbb{Y}$  be a dominant endomorphism. Then, there is a natural number  $N$  such that  $(\phi^N)^*$  is a diagonal matrix.*

**A.4. Endomorphisms on  $\mathbb{X}$ .**

**Lemma A.8.** *Let  $\phi$  is an endomorphism on  $\mathbb{X}$ . Then,  $\phi^*$  is an upper block-triangular matrix.*

$$\phi^* = \begin{pmatrix} A_1 & \cdots & B \\ O & \ddots & C \\ O & \cdots & A_s \end{pmatrix}$$

*Proof.* Let  $n_u < n_v$ . Suppose that  $\phi = (\phi_1, \dots, \phi_k)$  and  $\phi_u^* = (d_{u1}, \dots, d_{uk})$ . Then, Consider a morphism

$$\zeta = \phi_u \circ \iota_v : \mathbb{P}^{n_v} \rightarrow \mathbb{P}^{n_u}$$

where  $\iota_v : \mathbb{P}^{n_v} \hookrightarrow \mathbb{X}$  is the  $v$ -th closed embedding. Clearly, it is a morphism of degree  $d_{uv}$ . But it should be a constant map because of Lemma A.2 so that  $d_{uv} = 0$ .  $\square$

For convenience, we will also use another expression  $\mathbb{X} = \mathbb{Y}_1 \times \dots \times \mathbb{Y}_s$  where  $\mathbb{Y}_j = \mathbb{P}^{m_j} \times \dots \times \mathbb{P}^{m_j}$  and  $m_j < m_{j+1}$ .

**Theorem A.9.** *Let  $\phi$  is a dominant endomorphism on  $\mathbb{X}$ . Then,  $\phi^*$  is a block-diagonal matrix and hence*

$$\phi(P) = (\psi_1(P_1), \dots, \psi_s(P_s))$$

where  $\psi_j$  is a morphism on  $\mathbb{Y}_j$ . Furthermore, each diagonal block is nonsingular and multiplication of a permutation and a diagonal matrices.

*Proof.* Since  $\phi^*$  is an upper block-triangular matrix by Lemma A.8, it's enough to show all upper non-diagonal block is zero. Let  $n_u < n_v$ . Suppose that  $\phi_u^* = (d_{u1}, \dots, d_{uk})$  where  $d_{uv} \neq 0$ . Then, for any  $w$  satisfying  $n_w \geq n_v$ ,  $d_{uw} = 0$  because  $(\sum d_{ui} E_i)^{n_u+1} = 0$  guarantees

$$C d_{uv}^{m_u-1} d_{uv}^{n_v-n_u+1} E_u^{n_u} \cdot E_v^{n_v-n_u+1} = 0$$

while  $E_u^{n_u} \cdot E_v^{n_v-n_u+1} \neq 0$ .

So, a diagonal block  $A_j$  have a zero row and hence  $\phi^* D$  can't be ample for all ample divisor  $D$ . Thus  $\phi$  is not dominant and it's a contradiction. Therefore  $d_{uv} = 0$ . Which means  $\phi^*$  is a block-diagonal matrix:

$$\phi^* = \begin{pmatrix} \psi_1^* & O & O \\ O & \ddots & O \\ O & O & \psi_s^* \end{pmatrix}$$

which means

$$\phi(P) = (\psi_1, \dots, \psi_s)$$

where  $\psi_j$  is a dominant endomorphism on  $\mathbb{Y}_j$ .  $\square$

**Corollary A.10.** *Let  $\phi : \mathbb{X} \rightarrow \mathbb{X}$  be a dominant endomorphism. Then, for sufficiently large  $N$ ,  $\phi^N$  is a product of dominant endomorphism on  $\mathbb{P}^{n_i}$ ;*

$$\phi^N = \prod \phi_{N,i} : \mathbb{P}^{n_i} \rightarrow \mathbb{P}^{n_i}$$

Moreover,

$$\mu_1(\phi^N, D) = \mu(\phi^N) = \min \deg \phi_{N,i}, \quad \mu_2(\phi^N, D) = \max \deg \phi_{N,i}.$$

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